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A modified Heinz's inequality

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Abstract

Concerning the Heinz's inequality, Chan and Kwong [N.N. Chan, M.K. Kwong, Hermitian matrix inequalities and a conjecture, Amer. Math. Monthly 92 (1985) 533–541] conjectured that $A \geqslant B \geqslant O$ will imply $(BA^2B)^{\frac{1}{2}} \geqslant B^2$ and Furuta [T. Furuta, $A \geqslant B \geqslant O$ assures $(B^rA^pB^r)^{\frac{1}{q}} \geqslant B^{\frac{p+2r}{q}}$ for $r \geqslant 0$, $p \geqslant 0$, $q \geqslant 1$ with $(1+2r)q \geqslant p+2r$, Proc. Amer. Math. Soc. 101 (1987), 85–88] gave its affirmative answer as follows: If $A \geqslant B \geqslant O$, then $(B^rA^pB^r)^{\frac{1}{q}} \geqslant B^{\frac{p+2r}{q}}$, for $r \geqslant 0$, $p \geqslant 0$, $q \geqslant \max\left\{1, \frac{p+2r}{1+2r}\right\}$. And, in [K. Tanahashi, The Furuta inequality with negative powers, Proc. Amer. Math. Soc. 127 (1999) 1683–1692], Tanahashi studied the same inequality for the invertible case.

In this paper, we shall determine the region of γ such that the operator inequality $(A^\gamma A^\alpha A^\gamma)^\beta \geqslant (A^\gamma B^\alpha A^\gamma)^\beta$ holds for any bounded linear operators A and B on a Hilbert space $\mathscr{H}$ such as $A \geqslant B \geqslant bI$ (some $b > 0$) and for any given α and β such as $\alpha > 0$ and $\beta > 0$. It is easily seen that the inequalities $(A^\gamma A^\alpha A^\gamma)^\beta \geqslant (A^\gamma B^\alpha A^\gamma)^\beta$ and $(B^\gamma A^\alpha B^\gamma)^\beta \geqslant (B^\gamma B^\alpha B^\gamma)^\beta$ are equivalent.

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We use capital letters $A, B, \ldots$ to denote the bounded linear operators on the Hilbert space $\mathscr{H}$. The following results are well-known.

Proposition 1 (Heinz's inequality [2]). *$A \geqslant B \geqslant O$ implies that $A^\alpha \geqslant B^\alpha$, for all $\alpha \in [0, 1]$.*

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Proposition 2. *$A \geqslant B \geqslant bI$ (some $b > 0$) implies that $B^{-1} \geqslant A^{-1}$.*

Proposition 3. *If C is invertible, then $(C^*C)^\lambda = C^*(CC^*)^{\lambda-1}C$ for any real number λ.*

By Proposition 1, we have the following.

Lemma 1. *For $0 < \alpha \leqslant 1, 0 < \beta \leqslant 1, -\infty < \gamma < +\infty$ or for $0 < \alpha \leqslant 1, 0 < \beta \leqslant \frac{1}{\alpha}, \gamma = 0$, $A \geqslant B \geqslant bI$ (some $b > 0$) implies that $(A^\gamma A^\alpha A^\gamma)^\beta \geqslant (A^\gamma B^\alpha A^\gamma)^\beta$.*

Lemma 2 (cf. [3]). *For $0 < \alpha \leqslant 1, 1 < \beta \leqslant 2$, $\max\left\{-\frac{1}{2}, \frac{-\alpha\beta}{2(\beta-1)}\right\} \leqslant \gamma \leqslant \min\left\{0, \frac{1-\alpha\beta}{2(\beta-1)}\right\}$, $A \geqslant B \geqslant bI$ (some $b > 0$) implies $(A^\gamma A^\alpha A^\gamma)^\beta \geqslant (A^\gamma B^\alpha A^\gamma)^\beta$.*

Proof

$$
\begin{aligned}
(A^\gamma B^\alpha A^\gamma)^\beta &= A^\gamma B^{\frac{\alpha}{2}} \left(B^{\frac{\alpha}{2}} A^{2\gamma} B^{\frac{\alpha}{2}}\right)^{\beta-1} B^{\frac{\alpha}{2}} A^\gamma \quad \text{by Proposition 3} \\
&\leqslant A^\gamma B^{\frac{\alpha}{2}} \left(B^{\frac{\alpha}{2}} B^{2\gamma} B^{\frac{\alpha}{2}}\right)^{\beta-1} B^{\frac{\alpha}{2}} A^\gamma \quad \text{by Propositions 1 and 2} \\
&\quad \text{for } -1 \leqslant 2\gamma \leqslant 0 \\
&= A^\gamma B^{\alpha\beta+2\gamma(\beta-1)} A^\gamma \leqslant A^\gamma A^{\alpha\beta+2\gamma(\beta-1)} A^\gamma \quad \text{by Proposition 1} \\
&\quad \text{for } 0 \leqslant \alpha\beta + 2\gamma(\beta-1) \leqslant 1 \\
&= A^{\alpha\beta+2\gamma\beta} = (A^\gamma A^\alpha A^\gamma)^\beta. \qquad \square
\end{aligned}
$$

Lemma 3 (cf. [3]). *For $0 < \alpha \leqslant 1, 2 < \beta$, $\max\left\{\frac{2\alpha-1-\alpha\beta}{2(\beta-1)}, \frac{-\alpha\beta}{2(\beta-1)}\right\} \leqslant \gamma \leqslant \min\left\{\frac{2\alpha-\alpha\beta}{2(\beta-1)}, \frac{1-\alpha\beta}{2(\beta-1)}\right\}$, $A \geqslant B \geqslant bI$ (some $b > 0$) implies that $(A^\gamma A^\alpha A^\gamma)^\beta \geqslant (A^\gamma B^\alpha A^\gamma)^\beta$.*

Proof. If $2 < \beta \leqslant 3$, then we have

$$
\begin{aligned}
(A^\gamma B^\alpha A^\gamma)^\beta &= A^\gamma B^\alpha A^\gamma (A^\gamma B^\alpha A^\gamma)^{\beta-2} A^\gamma B^\alpha A^\gamma \\
&\leqslant A^\gamma B^\alpha A^\gamma (A^\gamma A^\alpha A^\gamma)^{\beta-2} A^\gamma B^\alpha A^\gamma \quad \text{by Proposition 1} \\
&= A^\gamma B^\alpha A^{\alpha(\beta-2)+2\gamma(\beta-1)} B^\alpha A^\gamma \\
&\leqslant A^\gamma B^\alpha B^{\alpha(\beta-2)+2\gamma(\beta-1)} B^\alpha A^\gamma \quad \text{by Propositions 1 and 2} \\
&\quad \text{for } -1 \leqslant \alpha(\beta-2) + 2\gamma(\beta-1) \leqslant 0 \\
&= A^\gamma B^{\alpha\beta+2\gamma(\beta-1)} A^\gamma \leqslant A^\gamma A^{\alpha\beta+2\gamma(\beta-1)} A^\gamma \quad \text{by Proposition 1} \\
&\quad \text{for } 0 \leqslant \alpha\beta + 2\gamma(\beta-1) \leqslant 1 \\
&= A^{\alpha\beta+2\gamma\beta} = (A^\gamma A^\alpha A^\gamma)^\beta.
\end{aligned}
$$

Next, assume that the assertion of Lemma 3 holds for the case $2 \leqslant n < \beta \leqslant n+1$ and let $n+1 < \beta \leqslant n+2$. Then we have

$$
\begin{aligned}
(A^\gamma B^\alpha A^\gamma)^\beta &= A^\gamma B^{\frac{\alpha}{2}} \left(B^{\frac{\alpha}{2}} A^{2\gamma} B^{\frac{\alpha}{2}}\right)^{\beta-1} B^{\frac{\alpha}{2}} A^\gamma \quad \text{by Proposition 3} \\
&= A^\gamma B^{\frac{\alpha}{2}} \left\{(B^{-1})^{-\frac{\alpha}{2}} (A^{-1})^{-2\gamma} (B^{-1})^{-\frac{\alpha}{2}}\right\}^{\beta-1} B^{\frac{\alpha}{2}} A^\gamma
\end{aligned}
$$

$$\begin{aligned}
&\leqslant A^\gamma B^{\frac{\alpha}{2}}\left\{(B^{-1})^{-\frac{\alpha}{2}}(B^{-1})^{-2\gamma}(B^{-1})^{-\frac{\alpha}{2}}\right\}^{\beta-1}B^{\frac{\alpha}{2}}A^\gamma \quad \text{by the assumption}\\
&\quad \text{for } 0<-2\gamma\leqslant 1 \quad \text{and}\\
&\qquad \max\left\{\frac{-4\gamma-1+2\gamma(\beta-1)}{2(\beta-2)},\frac{2\gamma(\beta-1)}{2(\beta-2)}\right\}\\
&\qquad\quad \leqslant -\frac{\alpha}{2}\leqslant \min\left\{\frac{-4\gamma+2\gamma(\beta-1)}{2(\beta-2)},\frac{1+2\gamma(\beta-1)}{2(\beta-2)}\right\}\\
&= A^\gamma B^{\alpha\beta+2\gamma(\beta-1)}A^\gamma \leqslant A^\gamma A^{\alpha\beta+2\gamma(\beta-1)}A^\gamma \quad \text{by Proposition 1}\\
&\qquad \text{for } 0\leqslant \alpha\beta+2\gamma(\beta-1)\leqslant 1\\
&= A^{\alpha\beta+2\gamma\beta}=(A^\gamma A^\alpha A^\gamma)^\beta.
\end{aligned}$$

Since

$$\frac{2\alpha-1-\alpha\beta}{2(\beta-1)}+\frac{1}{2}=\frac{(1-\alpha)(\beta-2)}{2(\beta-1)}\geqslant 0,$$

$$\frac{-\alpha\beta}{2(\beta-1)}+\frac{\alpha(\beta-2)}{2(\beta-3)}=\frac{\alpha}{(\beta-1)(\beta-3)}>0,$$

$$\frac{2\alpha-\alpha\beta}{2(\beta-1)}=\frac{-\alpha(\beta-2)}{2(\beta-1)}<0 \quad \text{and} \quad \frac{1-\alpha\beta}{2(\beta-1)}-\frac{1-\alpha(\beta-2)}{2(\beta-3)}=\frac{\alpha-1}{(\beta-1)(\beta-3)}\leqslant 0,$$

$$\max\left\{\frac{2\alpha-1-\alpha\beta}{2(\beta-1)},\frac{-\alpha\beta}{2(\beta-1)}\right\}\leqslant\gamma\leqslant\min\left\{\frac{2\alpha-\alpha\beta}{2(\beta-1)},\frac{1-\alpha\beta}{2(\beta-1)}\right\}$$

implies that

$$-\frac{1}{2}\leqslant\frac{2\alpha-1-\alpha\beta}{2(\beta-1)}\leqslant\gamma\leqslant\frac{2\alpha-\alpha\beta}{2(\beta-1)}<0 \quad \text{and}$$

$$-\frac{\alpha(\beta-2)}{2(\beta-3)}<\frac{-\alpha\beta}{2(\beta-1)}\leqslant\gamma\leqslant\frac{1-\alpha\beta}{2(\beta-1)}\leqslant\frac{1-\alpha(\beta-2)}{2(\beta-3)}$$

and hence we have

$$0<-2\gamma\leqslant 1,\quad \frac{2\gamma(\beta-1)}{2(\beta-2)}\leqslant-\frac{\alpha}{2}\leqslant\frac{1+2\gamma(\beta-1)}{2(\beta-2)} \quad \text{and}$$

$$\frac{-4\gamma-1+2\gamma(\beta-1)}{2(\beta-2)}\leqslant-\frac{\alpha}{2}<\frac{-4\gamma+2\gamma(\beta-1)}{2(\beta-2)}.$$

Therefore the assertion of Lemma 3 holds also for the case $n+1<\beta\leqslant n+2$ and, by the induction, the proof of Lemma 3 is completed. □

Lemma 4. *For* $1<\alpha, 0<\beta<1, \max\left\{0,\frac{\alpha\beta-1}{2(1-\beta)}\right\}\leqslant\gamma, A\geqslant B\geqslant bI$ (*some* $b>0$) *implies that* $(A^\gamma A^\alpha A^\gamma)^\beta\geqslant(A^\gamma B^\alpha A^\gamma)^\beta$.

Remark 1. This lemma is essentially same as Furuta inequality proved in [1].

Proof. Let $0 \leqslant \gamma \leqslant \frac{1}{2}$ and let $\beta = \frac{1+2\gamma}{\alpha+2\gamma}$. Then $0 < \beta < 1$ and

$$\begin{aligned}(A^\gamma B^\alpha A^\gamma)^\beta &= A^\gamma B^{\frac{\alpha}{2}} \left(B^{\frac{\alpha}{2}} A^{2\gamma} B^{\frac{\alpha}{2}}\right)^{\beta-1} B^{\frac{\alpha}{2}} A^\gamma \quad \text{by Proposition 3} \\ &\leqslant A^\gamma B^{\frac{\alpha}{2}} \left(B^{\frac{\alpha}{2}} B^{2\gamma} B^{\frac{\alpha}{2}}\right)^{\beta-1} B^{\frac{\alpha}{2}} A^\gamma \quad \text{by Propositions 1 and 2} \\ &= A^\gamma B^{(\alpha+2\gamma)\beta-2\gamma} A^\gamma = A^\gamma B A^\gamma \leqslant A^\gamma A A^\gamma \quad \text{by the assumption} \\ &= A^{1+2\gamma} = A^{(\alpha+2\gamma)\beta} = (A^\gamma A^\alpha A^\gamma)^\beta.\end{aligned}$$

Since $\frac{1}{\beta} > 1$, let $0 \leqslant \gamma_1 \leqslant \frac{1}{2}$ and let $\beta_1 = \frac{1+2\gamma_1}{\frac{1}{\beta}+2\gamma_1}$. Then $0 < \beta_1 < 1$ and, by applying the above result to the inequality $A^{1+2\gamma} \geqslant (A^\gamma B^\alpha A^\gamma)^\beta$ instead of $A \geqslant B$, we have

$$\left[(A^{1+2\gamma})^{\gamma_1}\{(A^\gamma B^\alpha A^\gamma)^\beta\}^{\frac{1}{\beta}}(A^{1+2\gamma})^{\gamma_1}\right]^{\beta_1} \leqslant (A^{1+2\gamma})^{1+2\gamma_1} \quad \text{and}$$
$$\{A^{\gamma+\gamma_1(1+2\gamma)} B^\alpha A^{\gamma+\gamma_1(1+2\gamma)}\}^{\beta_1} \leqslant A^{(1+2\gamma)(1+2\gamma_1)}.$$

Since $(1+2\gamma)(1+2\gamma_1) = 1 + 2\{\gamma + \gamma_1(1+2\gamma)\}$ and since

$$\beta_1 = \frac{1+2\gamma_1}{\frac{1}{\beta}+2\gamma_1} = \frac{1+2\gamma_1}{\frac{\alpha+2\gamma}{1+2\gamma}+2\gamma_1} = \frac{1+2\{\gamma+\gamma_1(1+2\gamma)\}}{\alpha+2\{\gamma+\gamma_1(1+2\gamma)\}},$$

we have $\{A^{\gamma+\gamma_1(1+2\gamma)} B^\alpha A^{\gamma+\gamma_1(1+2\gamma)}\}^{\beta_1} \leqslant \{A^{\gamma+\gamma_1(1+2\gamma)} A^\alpha A^{\gamma+\gamma_1(1+2\gamma)}\}^{\beta_1}$.

By repeating this argument, the operator inequality $(A^\gamma A^\alpha A^\gamma)^\beta \geqslant (A^\gamma B^\alpha A^\gamma)^\beta$ holds for the case where $\alpha > 1, \gamma \geqslant 0, \beta = \frac{1+2\gamma}{\alpha+2\gamma}$. By Proposition 1, it holds also for the case where $\alpha > 1, \gamma \geqslant 0, 0 < \beta \leqslant \frac{1+2\gamma}{\alpha+2\gamma}$.

Therefore it holds for the case where $\alpha > 1, 0 < \beta < 1, \gamma \geqslant \max\left\{0, \frac{\alpha\beta-1}{2(1-\beta)}\right\}$. □

Theorem. *The region of γ such that the operator inequality*

$$(A^\gamma A^\alpha A^\gamma)^\beta \geqslant (A^\gamma B^\alpha A^\gamma)^\beta$$

holds for any operators A and B such as $A \geqslant B \geqslant bI$ (some $b > 0$) and for any given α and β such as $\alpha > 0$ and $\beta > 0$ is as follows:

(1) $0 < \alpha \leqslant 1, 0 < \beta \leqslant 1$,

$-\infty < \gamma < +\infty$,

(2) $0 < \alpha \leqslant 1, 1 < \beta \leqslant 2$,

$$\max\left\{-\frac{1}{2}, \frac{-\alpha\beta}{2(\beta-1)}\right\} \leqslant \gamma \leqslant \min\left\{0, \frac{1-\alpha\beta}{2(\beta-1)}\right\},$$

(3) $0 < \alpha \leqslant 1, 2 < \beta \leqslant \frac{1}{\alpha}$,

$\gamma = 0$,

(4) $0 < \alpha \leqslant 1, 2 < \beta$,

$$\max\left\{\frac{2\alpha-1-\alpha\beta}{2(\beta-1)}, \frac{-\alpha\beta}{2(\beta-1)}\right\} \leqslant \gamma \leqslant \min\left\{\frac{2\alpha-\alpha\beta}{2(\beta-1)}, \frac{1-\alpha\beta}{2(\beta-1)}\right\}, \quad \textit{and}$$

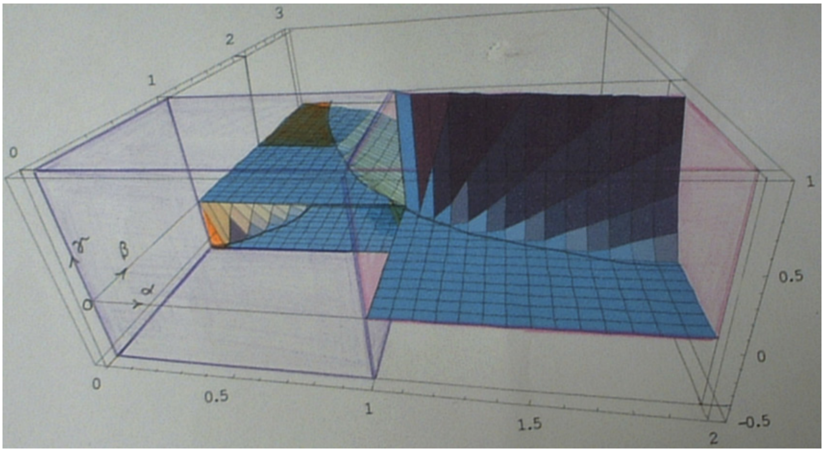

Fig. 1.

(5) $1 < \alpha, 0 < \beta < 1,$

$$\max\left\{0, \frac{\alpha\beta - 1}{2(1-\beta)}\right\} \leqslant \gamma.$$

Remark 2. An open question in [3] was solved negatively by our theorem because the range of the question is the following:

$$0 < \alpha < \frac{1}{2}, \quad 2 < \beta, \quad \frac{2\alpha - \alpha\beta}{2(\beta-1)} < \gamma \leqslant \frac{1-\alpha\beta}{2(\beta-1)} \quad \text{and} \quad -\frac{1}{4} < \gamma < 0.$$

Proof. By Lemmas 1–4, we have only to constitute counter examples of A and B in the outside of our ranges.

For a and b such as $a > 1 > b > 0$, let

$$x = x(a,b) = \frac{b\{(a-1) + a(1-b)\}}{a - 1 + b(1-b)} = by(a,b). \tag{$\sharp_1$}$$

Then

$$y = y(a,b) = \frac{(a-1) + a(1-b)}{a-1+b(1-b)} > 1 \quad \text{and} \quad \lim_{a\to\infty} y(a,b) = 2. \tag{$\sharp_2$}$$

For any ϵ such as $0 < \epsilon < \frac{(a-b)(a-1)}{b(1-b)}$, let $\delta = \frac{b(1-b)\epsilon}{a-1}$. Then $0 < \delta < a - b$.
And let

$$A = \begin{pmatrix} a & \sqrt{\epsilon(a-b-\delta)} \\ \sqrt{\epsilon(a-b-\delta)} & b+\epsilon+\delta \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} x & 0 \\ 0 & b \end{pmatrix}.$$

Then A and B are self-adjoint and $B \geqslant bI$ by ($\sharp_2$) and

$$\begin{aligned} a - x &= \frac{a^2 - a + ab - ab^2 - 2ab + b + ab^2}{a - 1 + b - b^2} \\ &= \frac{a^2 - a - ab + b}{a-1+b-b^2} = \frac{(a-1)(a-b)}{a-1+b(1-b)} > 0. \end{aligned}$$

Since the proper polynomial of $A - B = \begin{pmatrix} a-x & \sqrt{\epsilon(a-b-\delta)} \\ \sqrt{\epsilon(a-b-\delta)} & \epsilon+\delta \end{pmatrix}$ is

$$\lambda^2 - (a - x + \epsilon + \delta)\lambda + (a-x)(\epsilon+\delta) - \epsilon(a-b-\delta)$$

and since

$$\begin{aligned}(a-x)(\epsilon+\delta) - \epsilon(a-b-\delta) &= (a-x)(\epsilon+\delta) - \epsilon(a-b) + \epsilon\delta \\ &= \frac{\epsilon}{a-1}[(a-1)(a-x) + (a-x)b(1-b) - (a-1)(a-b)] + \epsilon\delta \\ &= \frac{\epsilon}{a-1}[a(a-1) + ab(1-b) - (a-1)(a-b) - x\{(a-1) + b(1-b)\}] + \epsilon\delta \\ &= \frac{\epsilon}{a-1}[b(2a-1-ab) - x(a-1+b-b^2)] + \epsilon\delta = \epsilon\delta > 0,\end{aligned}$$

we have $A \geqslant B$.

Since the proper equation of A is

$$\begin{aligned}0 &= \lambda^2 - (a+b+\epsilon+\delta)\lambda + a(b+\epsilon+\delta) - \epsilon(a-b-\delta) \\ &= \{\lambda - (a+\epsilon)\}\{\lambda - (b+\delta)\},\end{aligned}$$

its eigenvalues are $a+\epsilon$, $b+\delta$ and since their corresponding proper vectors are scalar multiples of $\begin{pmatrix} \sqrt{a-b-\delta} \\ \sqrt{\epsilon} \end{pmatrix}$, $\begin{pmatrix} \sqrt{\epsilon} \\ -\sqrt{a-b-\delta} \end{pmatrix}$ respectively,

$$U = \frac{1}{\sqrt{a-b-\delta+\epsilon}} \begin{pmatrix} \sqrt{a-b-\delta} & \sqrt{\epsilon} \\ \sqrt{\epsilon} & -\sqrt{a-b-\delta} \end{pmatrix}$$

is a self-adjoint unitary and $UAU = \begin{pmatrix} a+\epsilon & 0 \\ 0 & b+\delta \end{pmatrix}$.

Since

$$\begin{aligned}UB^{\alpha}U &= \frac{\begin{pmatrix} \sqrt{a-b-\delta} & \sqrt{\epsilon} \\ \sqrt{\epsilon} & -\sqrt{a-b-\delta} \end{pmatrix}\begin{pmatrix} x^{\alpha} & 0 \\ 0 & b^{\alpha} \end{pmatrix}\begin{pmatrix} \sqrt{a-b-\delta} & \sqrt{\epsilon} \\ \sqrt{\epsilon} & -\sqrt{a-b-\delta} \end{pmatrix}}{a-b-\delta+\epsilon} \\ &= \frac{1}{a-b-\delta+\epsilon}\begin{pmatrix} x^{\alpha}(a-b-\delta) + \epsilon b^{\alpha} & \sqrt{\epsilon(a-b-\delta)}(x^{\alpha}-b^{\alpha}) \\ \sqrt{\epsilon(a-b-\delta)}(x^{\alpha}-b^{\alpha}) & x^{\alpha}\epsilon + (a-b-\delta)b^{\alpha} \end{pmatrix},\end{aligned}$$

$$\begin{aligned}UA^{\gamma}B^{\alpha}A^{\gamma}U &= \frac{1}{a-b-\delta+\epsilon}\begin{pmatrix} (a+\epsilon)^{\gamma} & 0 \\ 0 & (b+\delta)^{\gamma} \end{pmatrix} \\ &\quad \times \begin{pmatrix} x^{\alpha}(a-b-\delta) + \epsilon b^{\alpha} & \sqrt{\epsilon(a-b-\delta)}(x^{\alpha}-b^{\alpha}) \\ \sqrt{\epsilon(a-b-\delta)}(x^{\alpha}-b^{\alpha}) & x^{\alpha}\epsilon + (a-b-\delta)b^{\alpha} \end{pmatrix} \\ &\quad \times \begin{pmatrix} (a+\epsilon)^{\gamma} & 0 \\ 0 & (b+\delta)^{\gamma} \end{pmatrix} = \frac{1}{a-b-\delta+\epsilon}\begin{pmatrix} c_1 & c_3 \\ c_3 & c_2 \end{pmatrix},\end{aligned}$$

where

$$\begin{aligned}c_1 &= (a+\epsilon)^{2\gamma}\{x^{\alpha}(a-b-\delta) + \epsilon b^{\alpha}\}, \\ c_2 &= (b+\delta)^{2\gamma}\{x^{\alpha}\epsilon + (a-b-\delta)b^{\alpha}\} \quad \text{and} \\ c_3 &= (a+\epsilon)^{\gamma}(b+\delta)^{\gamma}\sqrt{\epsilon(a-b-\delta)}(x^{\alpha}-b^{\alpha}).\end{aligned}$$

And we have,

$$
\begin{aligned}
c_1 &= a^{2\gamma}\left(1+\frac{\epsilon}{a}\right)^{2\gamma} x^\alpha(a-b)\left(1-\frac{\delta}{a-b}+\frac{\epsilon b^\alpha}{x^\alpha(a-b)}\right)\\
&= a^{2\gamma}x^\alpha(a-b)\left[1+\left\{\frac{2\gamma}{a}-\frac{b(1-b)}{(a-b)(a-1)}+\frac{b^\alpha}{x^\alpha(a-b)}\right\}\epsilon+\mathrm{o}(\epsilon)\right],\\
c_2 &= b^{2\gamma}\left(1+\frac{\delta}{b}\right)^{2\gamma}(a-b)b^\alpha\left\{1-\frac{\delta}{a-b}+\frac{x^\alpha\epsilon}{(a-b)b^\alpha}\right\}\\
&= b^{\alpha+2\gamma}(a-b)\left[1+\left\{\left(\frac{2\gamma}{b}-\frac{1}{a-b}\right)\frac{b(1-b)}{a-1}+\frac{x^\alpha}{(a-b)b^\alpha}\right\}\epsilon+\mathrm{o}(\epsilon)\right],\\
c_3^2 &= a^{2\gamma}\left(1+\frac{\epsilon}{a}\right)^{2\gamma} b^{2\gamma}\left(1+\frac{\delta}{b}\right)^{2\gamma}(x^\alpha-b^\alpha)^2\epsilon(a-b)\left(1-\frac{\delta}{a-b}\right)\\
&= a^{2\gamma}b^{2\gamma}(x^\alpha-b^\alpha)^2(a-b)\epsilon\left\{1+\frac{\mathrm{o}(\epsilon)}{\epsilon}\right\},
\end{aligned}
$$

where $\mathrm{o}(\epsilon)$ is a function of ϵ such that $\lim_{\epsilon\to 0}\frac{\mathrm{o}(\epsilon)}{\epsilon}=0$.

Since the proper equation of $\begin{pmatrix} c_1 & c_3\\ c_3 & c_2\end{pmatrix}$ is $0=\lambda^2-(c_1+c_2)\lambda+c_1c_2-c_3^2$, its eigenvalues are

$$
\lambda_1=\frac{c_1+c_2+\sqrt{(c_1-c_2)^2+4c_3^2}}{2}\quad\text{and}\quad \lambda_2=\frac{c_1+c_2-\sqrt{(c_1-c_2)^2+4c_3^2}}{2}.
$$

In the case where $c_1-c_2<0$, let

$$
s=\frac{c_1-c_2+\sqrt{(c_1-c_2)^2+4c_3^2}}{2}.
$$

Then $s>0$, $c_1-\lambda_2=\lambda_1-c_2=s$, $2s+c_2-c_1=\sqrt{(c_1-c_2)^2+4c_3^2}>0$ and

$$
\begin{aligned}
s&=\frac{1}{2}(c_2-c_1)\left\{-1+\sqrt{1+\frac{4c_3^2}{(c_2-c_1)^2}}\right\}\\
&=\frac{1}{2}(c_2-c_1)\left[-1+\left\{1+\frac{2c_3^2}{(c_2-c_1)^2}+\mathrm{o}(\epsilon)\right\}\right]=\frac{c_3^2}{c_2-c_1}+\mathrm{o}(\epsilon)\\
&=\frac{a^{2\gamma}b^{2\gamma}(x^\alpha-b^\alpha)^2\epsilon}{b^{\alpha+2\gamma}-a^{2\gamma}x^\alpha}\left\{1+\frac{\mathrm{o}(\epsilon)}{\epsilon}\right\}.
\end{aligned}
$$

And $c_3^2=s(s+c_2-c_1)$ implies $s+c_2-c_1>0$. Since the corresponding proper vectors of λ_1,λ_2 are scalar multiples of $\binom{s}{c_3}$, $\binom{c_3}{-s}$ respectively,

$$
V=\frac{1}{\sqrt{s^2+c_3^2}}\begin{pmatrix} s & c_3\\ c_3 & -s\end{pmatrix}=\frac{1}{\sqrt{2s+c_2-c_1}}\begin{pmatrix} \sqrt{s} & \sqrt{s+c_2-c_1}\\ \sqrt{s+c_2-c_1} & -\sqrt{s}\end{pmatrix}
$$

is a self-adjoint unitary and $V\begin{pmatrix} c_1 & c_3 \\ c_3 & c_2 \end{pmatrix}V = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$ and hence

$$\begin{aligned}\begin{pmatrix} c_1 & c_3 \\ c_3 & c_2 \end{pmatrix}^\beta &= V\begin{pmatrix} {\lambda_1}^\beta & 0 \\ 0 & {\lambda_2}^\beta \end{pmatrix}V \\ &= \frac{1}{2s+c_2-c_1}\begin{pmatrix} s{\lambda_1}^\beta+(s+c_2-c_1){\lambda_2}^\beta & \sqrt{s(s+c_2-c_1)}({\lambda_1}^\beta-{\lambda_2}^\beta) \\ \sqrt{s(s+c_2-c_1)}({\lambda_1}^\beta-{\lambda_2}^\beta) & (s+c_2-c_1){\lambda_1}^\beta+s{\lambda_2}^\beta \end{pmatrix}.\end{aligned}$$

Since $U(A^\gamma B^\alpha A^\gamma)^\beta U = \{U(A^\gamma B^\alpha A^\gamma)U\}^\beta$, we have

$$\begin{aligned}U(A^\gamma B^\alpha A^\gamma)^\beta U &= \frac{1}{(a-b-\delta+\epsilon)^\beta}\begin{pmatrix} c_1 & c_3 \\ c_3 & c_2 \end{pmatrix}^\beta \\ &= \frac{\begin{pmatrix} s{\lambda_1}^\beta+(s+c_2-c_1){\lambda_2}^\beta & \sqrt{s(s+c_2-c_1)}({\lambda_1}^\beta-{\lambda_2}^\beta) \\ \sqrt{s(s+c_2-c_1)}({\lambda_1}^\beta-{\lambda_2}^\beta) & (s+c_2-c_1){\lambda_1}^\beta+s{\lambda_2}^\beta \end{pmatrix}}{(a-b-\delta+\epsilon)^\beta(2s+c_2-c_1)}.\end{aligned}$$

If $(A^\gamma A^\alpha A^\gamma)^\beta \geqslant (A^\gamma B^\alpha A^\gamma)^\beta$, then $U(A^\gamma A^\alpha A^\gamma)^\beta U - U(A^\gamma B^\alpha A^\gamma)^\beta U \geqslant O$ and this implies that

$$\begin{aligned}0 &\leqslant \left\{(a+\epsilon)^{(\alpha+2\gamma)\beta} - \frac{s{\lambda_1}^\beta+(s+c_2-c_1){\lambda_2}^\beta}{(a-b-\delta+\epsilon)^\beta(2s+c_2-c_1)}\right\} \\ &\quad \times \left\{(b+\delta)^{(\alpha+2\gamma)\beta} - \frac{(s+c_2-c_1){\lambda_1}^\beta+s{\lambda_2}^\beta}{(a-b-\delta+\epsilon)^\beta(2s+c_2-c_1)}\right\} \\ &\quad - \frac{s(s+c_2-c_1)({\lambda_1}^\beta-{\lambda_2}^\beta)^2}{(a-b-\delta+\epsilon)^{2\beta}(2s+c_2-c_1)^2} \\ &= (a+\epsilon)^{(\alpha+2\gamma)\beta}(b+\delta)^{(\alpha+2\gamma)\beta} + \frac{{\lambda_1}^\beta{\lambda_2}^\beta}{h^{2\beta}} \\ &\quad - \frac{(a+\epsilon)^{(\alpha+2\gamma)\beta}\{t{\lambda_1}^\beta+s{\lambda_2}^\beta\} + (b+\delta)^{(\alpha+2\gamma)\beta}\{s{\lambda_1}^\beta+t{\lambda_2}^\beta\}}{h^\beta(s+t)} \\ &= \frac{s}{s+t}\left\{(a+\epsilon)^{(\alpha+2\gamma)\beta} - \frac{{\lambda_1}^\beta}{h^\beta}\right\}\left\{(b+\delta)^{(\alpha+2\gamma)\beta} - \frac{{\lambda_2}^\beta}{h^\beta}\right\} \\ &\quad + \frac{t}{s+t}\left\{(a+\epsilon)^{(\alpha+2\gamma)\beta} - \frac{{\lambda_2}^\beta}{h^\beta}\right\}\left\{(b+\delta)^{(\alpha+2\gamma)\beta} - \frac{{\lambda_1}^\beta}{h^\beta}\right\},\end{aligned}$$

where $t = s+c_2-c_1$ and $h = a-b-\delta+\epsilon$.

Since $s(s+t) = s(2s+c_2-c_1) > 0$ and since $st = s(s+c_2-c_1) = c_3^2$, we have

$$\begin{aligned}&c_3^2\left\{\frac{{\lambda_2}^\beta}{h^\beta} - (a+\epsilon)^{(\alpha+2\gamma)\beta}\right\}\left\{(b+\delta)^{(\alpha+2\gamma)\beta} - \frac{{\lambda_1}^\beta}{h^\beta}\right\} \\ &\quad \leqslant s^2\left\{(a+\epsilon)^{(\alpha+2\gamma)\beta} - \frac{{\lambda_1}^\beta}{h^\beta}\right\}\left\{(b+\delta)^{(\alpha+2\gamma)\beta} - \frac{{\lambda_2}^\beta}{h^\beta}\right\}\end{aligned}$$

and

$$c_3^2\left\{\left(\frac{c_1-s}{a-b-\delta+\epsilon}\right)^\beta - (a+\epsilon)^{(\alpha+2\gamma)\beta}\right\}\left\{(b+\delta)^{(\alpha+2\gamma)\beta} - \left(\frac{c_2+s}{a-b-\delta+\epsilon}\right)^\beta\right\}$$

$$\leqslant s^2\left\{(a+\epsilon)^{(\alpha+2\gamma)\beta}-\left(\frac{c_2+s}{a-b-\delta+\epsilon}\right)^{\beta}\right\}\left\{(b+\delta)^{(\alpha+2\gamma)\beta}-\left(\frac{c_1-s}{a-b-\delta+\epsilon}\right)^{\beta}\right\}. \tag{i}$$

Since

$$\begin{aligned}
c_1-s&=a^{2\gamma}x^{\alpha}(a-b)\left[1+\left\{\frac{2\gamma}{a}-\frac{b(1-b)}{(a-b)(a-1)}+\frac{b^{\alpha}}{x^{\alpha}(a-b)}\right.\right.\\
&\qquad\left.\left.+\frac{b^{2\gamma}(x^{\alpha}-b^{\alpha})^2}{(a-b)x^{\alpha}(a^{2\gamma}x^{\alpha}-b^{\alpha+2\gamma})}\right\}\epsilon+\mathrm{o}(\epsilon)\right],\\
c_2+s&=b^{\alpha+2\gamma}(a-b)\left[1+\left\{\left(\frac{2\gamma}{b}-\frac{1}{a-b}\right)\frac{b(1-b)}{a-1}+\frac{x^{\alpha}}{(a-b)b^{\alpha}}\right.\right.\\
&\qquad\left.\left.-\frac{a^{2\gamma}(x^{\alpha}-b^{\alpha})^2}{(a-b)b^{\alpha}(a^{2\gamma}x^{\alpha}-b^{\alpha+2\gamma})}\right\}\epsilon+\mathrm{o}(\epsilon)\right],
\end{aligned}$$

$$\begin{aligned}
&\left(\frac{c_1-s}{a-b-\delta+\epsilon}\right)^{\beta}\\
&\quad=a^{2\gamma\beta}x^{\alpha\beta}\left[1+\left\{\frac{2\gamma}{a}+\frac{b^{\alpha}-x^{\alpha}}{x^{\alpha}(a-b)}+\frac{b^{2\gamma}(x^{\alpha}-b^{\alpha})^2}{(a-b)x^{\alpha}(a^{2\gamma}x^{\alpha}-b^{\alpha+2\gamma})}\right\}\beta\epsilon+\mathrm{o}(\epsilon)\right],\\
&\left(\frac{c_2+s}{a-b-\delta+\epsilon}\right)^{\beta}\\
&\quad=b^{(\alpha+2\gamma)\beta}\left[1+\left\{\frac{2\gamma(1-b)}{a-1}+\frac{x^{\alpha}-b^{\alpha}}{(a-b)b^{\alpha}}-\frac{a^{2\gamma}(x^{\alpha}-b^{\alpha})^2}{(a-b)b^{\alpha}(a^{2\gamma}x^{\alpha}-b^{\alpha+2\gamma})}\right\}\beta\epsilon+\mathrm{o}(\epsilon)\right],
\end{aligned}$$

$$\begin{aligned}
(a+\epsilon)^{(\alpha+2\gamma)\beta}&=a^{(\alpha+2\gamma)\beta}\left\{1+\frac{(\alpha+2\gamma)\beta\epsilon}{a}+\mathrm{o}(\epsilon)\right\},\\
(b+\delta)^{(\alpha+2\gamma)\beta}&=b^{(\alpha+2\gamma)\beta}\left\{1+\frac{(1-b)(\alpha+2\gamma)\beta\epsilon}{a-1}+\mathrm{o}(\epsilon)\right\},
\end{aligned}$$

we have

$$\begin{aligned}
&a^{2\gamma}b^{2\gamma}(x^{\alpha}-b^{\alpha})^2(a-b)\epsilon\left\{1+\frac{\mathrm{o}(\epsilon)}{\epsilon}\right\}\\
&\qquad\times\left[\{a^{2\gamma\beta}x^{\alpha\beta}-a^{(\alpha+2\gamma)\beta}\}+a^{2\gamma\beta}\left\{\frac{2\gamma(x^{\alpha\beta}-a^{\alpha\beta})}{a}+\frac{x^{\alpha\beta}(b^{\alpha}-x^{\alpha})}{x^{\alpha}(a-b)}\right.\right.\\
&\qquad\left.\left.+\frac{x^{\alpha\beta}b^{2\gamma}(x^{\alpha}-b^{\alpha})^2}{(a-b)x^{\alpha}(a^{2\gamma}x^{\alpha}-b^{\alpha+2\gamma})}-\alpha a^{\alpha\beta-1}\right\}\beta\epsilon+\mathrm{o}(\epsilon)\right]\\
&\qquad\times b^{(\alpha+2\gamma)\beta}\left[\left\{\frac{\alpha(1-b)}{a-1}-\frac{x^{\alpha}-b^{\alpha}}{(a-b)b^{\alpha}}+\frac{a^{2\gamma}(x^{\alpha}-b^{\alpha})^2}{(a-b)b^{\alpha}(a^{2\gamma}x^{\alpha}-b^{\alpha+2\gamma})}\right\}\beta\epsilon+\mathrm{o}(\epsilon)\right]\\
&\quad\leqslant\frac{a^{4\gamma}b^{4\gamma}(x^{\alpha}-b^{\alpha})^4\epsilon^2}{(a^{2\gamma}x^{\alpha}-b^{\alpha+2\gamma})^2}\left\{1+\frac{\mathrm{o}(\epsilon)}{\epsilon}\right\}\left[\{a^{(\alpha+2\gamma)\beta}-b^{(\alpha+2\gamma)\beta}\}\right.
\end{aligned}$$

$$
\begin{aligned}
&+\left\{(\alpha+2\gamma)a^{(\alpha+2\gamma)\beta-1}-\left(\frac{2\gamma(1-b)}{a-1}+\frac{x^\alpha-b^\alpha}{(a-b)b^\alpha}\right.\right.\\
&\left.\left.-\frac{a^{2\gamma}(x^\alpha-b^\alpha)^2}{(a-b)b^\alpha(a^{2\gamma}x^\alpha-b^{\alpha+2\gamma})}\right)b^{(\alpha+2\gamma)\beta}\right\}\beta\epsilon+\mathrm{o}(\epsilon)\Bigg]\\
&\times\Bigg[\{b^{(\alpha+2\gamma)\beta}-a^{2\gamma\beta}x^{\alpha\beta}\}+\left\{\frac{(\alpha+2\gamma)(1-b)b^{(\alpha+2\gamma)\beta}}{a-1}\right.\\
&\left.-\left(\frac{2\gamma}{a}+\frac{b^\alpha-x^\alpha}{x^\alpha(a-b)}+\frac{b^{2\gamma}(x^\alpha-b^\alpha)^2}{(a-b)x^\alpha(a^{2\gamma}x^\alpha-b^{\alpha+2\gamma})}\right)a^{2\gamma\beta}x^{\alpha\beta}\right\}\beta\epsilon+\mathrm{o}(\epsilon)\Bigg]
\end{aligned}
$$

by (i) and, by multiplying $\{a^{2\gamma(1+\beta)}b^{4\gamma}(x^\alpha-b^\alpha)^4\epsilon^2\}^{-1}$ to the both side of the above inequality and by putting $\epsilon\to 0$, we have

$$
\begin{aligned}
&\frac{\beta(a^{\alpha\beta}-x^{\alpha\beta})b^{(\alpha+2\gamma)\beta-2\gamma}}{(x^\alpha-b^\alpha)^2}\left\{\frac{x^\alpha-b^\alpha}{b^\alpha}-\frac{\alpha(1-b)(a-b)}{a-1}-\frac{a^{2\gamma}(x^\alpha-b^\alpha)^2}{b^\alpha(a^{2\gamma}x^\alpha-b^{\alpha+2\gamma})}\right\}\\
&\leqslant\frac{a^{2\gamma(1-\beta)}}{(b^{\alpha+2\gamma}-a^{2\gamma}x^\alpha)^2}\{a^{(\alpha+2\gamma)\beta}-b^{(\alpha+2\gamma)\beta}\}\{b^{(\alpha+2\gamma)\beta}-a^{2\gamma\beta}x^{\alpha\beta}\}. \qquad \text{(ii)}
\end{aligned}
$$

We remark here that, by putting $s=\frac{c_2-c_1+\sqrt{(c_1-c_2)^2+4c_3^2}}{2}$ in the case where $c_1-c_2>0$ and by using the same argument as above, we have the same inequality (ii).

Since $x=by$ by ($\sharp_1$), by multiplying $b^{2\alpha-\alpha\beta}(a^{\alpha\beta}-x^{\alpha\beta})^{-1}$ to the both side of (ii), we have

$$
\begin{aligned}
&\frac{\beta b^{2\gamma(\beta-1)}}{(y^\alpha-1)^2}\left\{y^\alpha-1-\frac{\alpha(1-b)(a-b)}{(a-1)}-\frac{a^{2\gamma}(y^\alpha-1)^2}{a^{2\gamma}y^\alpha-b^{2\gamma}}\right\}\\
&\leqslant\frac{a^{2\gamma(1-\beta)}}{(a^{\alpha\beta}-b^{\alpha\beta}y^{\alpha\beta})(b^{2\gamma}-a^{2\gamma}y^\alpha)^2}\{a^{(\alpha+2\gamma)\beta}-b^{(\alpha+2\gamma)\beta}\}\{b^{2\gamma\beta}-a^{2\gamma\beta}y^{\alpha\beta}\} \qquad \text{(iii)}\\
&=\frac{1}{(1-a^{-\alpha\beta}b^{\alpha\beta}y^{\alpha\beta})(a^{-2\gamma}b^{2\gamma}-y^\alpha)^2}\{a^{2\gamma(\beta-1)}-a^{-(\alpha\beta+2\gamma)}b^{(\alpha+2\gamma)\beta}\}\\
&\quad\times\{a^{-2\gamma\beta}b^{2\gamma\beta}-y^{\alpha\beta}\} \qquad \text{(iii}'\text{)}\\
&=\frac{a^{2\gamma}}{(1-a^{-\alpha\beta}b^{\alpha\beta}y^{\alpha\beta})(b^{2\gamma}-a^{2\gamma}y^\alpha)^2}\{1-a^{-(\alpha+2\gamma)\beta}b^{(\alpha+2\gamma)\beta}\}\\
&\quad\times\{b^{2\gamma\beta}-a^{2\gamma\beta}y^{\alpha\beta}\} \qquad \text{(iii}''\text{)}\\
&=\frac{a^{2\gamma(1-\beta)-\alpha\beta}}{(1-a^{-\alpha\beta}b^{\alpha\beta}y^{\alpha\beta})(b^{2\gamma}-a^{2\gamma}y^\alpha)^2}\{a^{(\alpha+2\gamma)\beta}-b^{(\alpha+2\gamma)\beta}\}\\
&\quad\times\{b^{2\gamma\beta}-a^{2\gamma\beta}y^{\alpha\beta}\}. \qquad \text{(iii}'''\text{)}
\end{aligned}
$$

Case 1. Let $0 < \alpha$, $1 < \beta$, $0 < \gamma$.

Then

$$\begin{aligned}&\lim_{a\to\infty}\frac{\beta b^{2\gamma(\beta-1)}}{(y^\alpha-1)^2}\left\{y^\alpha-1-\frac{\alpha(1-b)(a-b)}{(a-1)}-\frac{a^{2\gamma}(y^\alpha-1)^2}{a^{2\gamma}y^\alpha-b^{2\gamma}}\right\}\\&\quad=\frac{\beta b^{2\gamma(\beta-1)}}{(2^\alpha-1)^2}\left\{2^\alpha-1-\alpha(1-b)-\frac{(2^\alpha-1)^2}{2^\alpha}\right\}\end{aligned}$$

because $\lim_{a\to\infty} y = 2$ by ($\sharp_2$) and

$$\begin{aligned}&\lim_{a\to\infty}\frac{1}{(1-a^{-\alpha\beta}b^{\alpha\beta}y^{\alpha\beta})(a^{-2\gamma}b^{2\gamma}-y^\alpha)^2}\{a^{2\gamma(\beta-1)}-a^{-(\alpha\beta+2\gamma)}b^{(\alpha+2\gamma)\beta}\}\\&\quad\times\{a^{-2\gamma\beta}b^{2\gamma\beta}-y^{\alpha\beta}\}=-\infty.\end{aligned}$$

This contradicts (iii′).

Case 2. Let $1 < \alpha$, $0 < \beta$, $\gamma < 0$.

Then

$$\begin{aligned}&\lim_{a\to\infty}\frac{\beta b^{2\gamma(\beta-1)}}{(y^\alpha-1)^2}\left\{y^\alpha-1-\frac{\alpha(1-b)(a-b)}{(a-1)}-\frac{a^{2\gamma}(y^\alpha-1)^2}{a^{2\gamma}y^\alpha-b^{2\gamma}}\right\}\\&\quad=\frac{\beta b^{2\gamma(\beta-1)}}{(2^\alpha-1)^2}\{2^\alpha-1-\alpha(1-b)\}\end{aligned}$$

because $\lim_{a\to\infty} y = 2$ by ($\sharp_2$).

If $\alpha + 2\gamma \geqslant 0$, then we have

$$\begin{aligned}&\lim_{a\to\infty}\frac{a^{2\gamma}}{(1-a^{-\alpha\beta}b^{\alpha\beta}y^{\alpha\beta})(b^{2\gamma}-a^{2\gamma}y^\alpha)^2}\{1-a^{-(\alpha+2\gamma)\beta}b^{(\alpha+2\gamma)\beta}\}\\&\quad\times\{b^{2\gamma\beta}-a^{2\gamma\beta}y^{\alpha\beta}\}=0\end{aligned}$$

and, if $\alpha + 2\gamma < 0$, then we have also

$$\begin{aligned}&\lim_{a\to\infty}\frac{a^{2\gamma(1-\beta)-\alpha\beta}}{(1-a^{-\alpha\beta}b^{\alpha\beta}y^{\alpha\beta})(b^{2\gamma}-a^{2\gamma}y^\alpha)^2}\{a^{(\alpha+2\gamma)\beta}-b^{(\alpha+2\gamma)\beta}\}\{b^{2\gamma\beta}-a^{2\gamma\beta}y^{\alpha\beta}\}\\&\quad=\begin{cases}0, & (2\gamma(1-\beta)-\alpha\beta<0),\\ -b^{-\alpha\beta}, & (2\gamma(1-\beta)-\alpha\beta=0),\\ -\infty, & (2\gamma(1-\beta)-\alpha\beta>0),\end{cases}\quad\leqslant 0\end{aligned}$$

and hence, by (iii″) and (iii‴), we have $2^\alpha - 1 - \alpha(1-b) \leqslant 0$ and

$$b \leqslant \frac{\alpha+1-2^\alpha}{\alpha}.$$

Since $\alpha + 1 - 2^\alpha < 0$ for all $\alpha > 1$, this contradicts $b > 0$.

Case 3. Let $0 < \alpha$, $\frac{1}{\alpha} < \beta$, $\gamma = 0$.

Then

$$\lim_{a\to\infty}\frac{\beta b^{2\gamma(\beta-1)}}{(y^\alpha-1)^2}\left\{y^\alpha-1-\frac{\alpha(1-b)(a-b)}{(a-1)}-\frac{a^{2\gamma}(y^\alpha-1)^2}{a^{2\gamma}y^\alpha-b^{2\gamma}}\right\}=\frac{-\alpha\beta(1-b)}{(2^\alpha-1)^2}$$

and

$$\lim_{a\to\infty} \frac{1}{(1-a^{-\alpha\beta}b^{\alpha\beta}y^{\alpha\beta})(a^{-2\gamma}b^{2\gamma}-y^{\alpha})^2}\{a^{2\gamma(\beta-1)} - a^{-(\alpha\beta+2\gamma)}b^{(\alpha+2\gamma)\beta}\}$$
$$\times \{a^{-2\gamma\beta}b^{2\gamma\beta} - y^{\alpha\beta}\} = \frac{1-2^{\alpha\beta}}{(1-2^{\alpha})^2}$$

because $\lim_{a\to\infty} y = 2$ by ($\sharp_2$).

By (iii$'$), we have $\alpha\beta(b-1) \leqslant 1-2^{\alpha\beta}$ and

$$b \leqslant \frac{\alpha\beta+1-2^{\alpha\beta}}{\alpha\beta}.$$

Since $\alpha\beta+1-2^{\alpha\beta} < 0$ for all $\alpha\beta > 1$, this contradicts $b > 0$.

Case 4. Let $0 < \alpha \leqslant 1$, $1 < \beta$, $\min\left\{\min\left(0, \frac{2\alpha-\alpha\beta}{2(\beta-1)}\right), \frac{1-\alpha\beta}{2(\beta-1)}\right\} < \gamma < 0$.

In this case

$$\begin{cases} (\alpha+2\gamma)\beta - 2\gamma - \alpha - \min\{\min(\alpha(\beta-1),\alpha), 1-\alpha\} \\ \quad = 2\gamma(\beta-1) + \alpha\beta - \alpha - \min\{\min(\alpha(\beta-1),\alpha), 1-\alpha\} > 0 \quad \text{and} \\ (\alpha+2\gamma)(\beta-1) = (\alpha+2\gamma)\beta - 2\gamma - \alpha > \min\{\min(\alpha(\beta-1),\alpha),1-\alpha\} \geqslant 0 \\ \quad \text{and hence } \alpha+2\gamma > 0 \text{ because } \beta > 1. \end{cases}$$

Then we have

$$\lim_{a\to\infty} \frac{\beta b^{2\gamma(\beta-1)}}{(y^{\alpha}-1)^2}\left\{y^{\alpha} - 1 - \frac{\alpha(1-b)(a-b)}{(a-1)} - \frac{a^{2\gamma}(y^{\alpha}-1)^2}{a^{2\gamma}y^{\alpha}-b^{2\gamma}}\right\}$$
$$= \frac{\beta b^{2\gamma(\beta-1)}}{(2^{\alpha}-1)^2}\{2^{\alpha}-1-\alpha(1-b)\}$$

because $\lim_{a\to\infty} y = 2$ by ($\sharp_2$) and

$$\lim_{a\to\infty} \frac{a^{2\gamma}}{(1-a^{-\alpha\beta}b^{\alpha\beta}y^{\alpha\beta})(b^{2\gamma}-a^{2\gamma}y^{\alpha})^2}\{1 - a^{-(\alpha+2\gamma)\beta}b^{(\alpha+2\gamma)\beta}\}$$
$$\times \{b^{2\gamma\beta} - a^{2\gamma\beta}y^{\alpha\beta}\} = 0.$$

And, by (iii$''$), we have $2^{\alpha}-1-\alpha(1-b) \leqslant 0$ and

$$b \leqslant \frac{\alpha+1-2^{\alpha}}{\alpha}.$$

Since $\frac{\alpha+1-2^{\alpha}}{\alpha} < 1-\log 2 \fallingdotseq 0.306853$ for all α such as $0 < \alpha \leqslant 1$ and since we may take firstly $b = \frac{1}{2}$, this is a contradiction.

Case 5. Let $0 < \alpha \leqslant 1$, $1 < \beta$, $\gamma < \max\left\{\max\left(-\frac{1}{2}, \frac{2\alpha-1-\alpha\beta}{2(\beta-1)}\right), \frac{-\alpha\beta}{2(\beta-1)}\right\}$.

In this case

$$\begin{cases} 2\gamma(\beta-1) + \alpha\beta + 1 - \alpha - \max\{\max(\alpha\beta+2-\beta-\alpha, \alpha), 1-\alpha\} < 0 \quad \text{and} \\ (\alpha+2\gamma)(\beta-1) < -1 + \max\{\max(\alpha\beta+2-\beta-\alpha,\alpha), 1-\alpha\} \\ \quad = -\min\{(1-\alpha)\min(\beta-1,1), \alpha\} \leqslant 0 \\ \text{and hence } \alpha+2\gamma < 0 \text{ because } \beta > 1. \end{cases}$$

Then we have

$$\lim_{a\to\infty} \frac{\beta b^{2\gamma(\beta-1)}}{(y^\alpha-1)^2}\left\{y^\alpha - 1 - \frac{\alpha(1-b)(a-b)}{(a-1)} - \frac{a^{2\gamma}(y^\alpha-1)^2}{a^{2\gamma}y^\alpha - b^{2\gamma}}\right\}$$
$$= \frac{\beta b^{2\gamma(\beta-1)}}{(2^\alpha-1)^2}\{2^\alpha - 1 - \alpha(1-b)\}$$

because $\lim_{a\to\infty} y = 2$ by ($\sharp_2$) and

$$\lim_{a\to\infty} \frac{a^{2\gamma(1-\beta)-\alpha\beta}}{(1-a^{-\alpha\beta}b^{\alpha\beta}y^{\alpha\beta})(b^{2\gamma}-a^{2\gamma}y^\alpha)^2}\{a^{(\alpha+2\gamma)\beta} - b^{(\alpha+2\gamma)\beta}\}\{b^{2\gamma\beta} - a^{2\gamma\beta}y^{\alpha\beta}\}$$
$$= \begin{cases} 0, & (2\gamma(1-\beta)-\alpha\beta < 0), \\ -b^{-\alpha\beta}, & (2\gamma(1-\beta)-\alpha\beta = 0), \\ -\infty, & (2\gamma(1-\beta)-\alpha\beta > 0). \end{cases}$$

And, by (iii$'''$), we have $2^\alpha - 1 - \alpha(1-b) \leqslant 0$ and, by the same reason as in the (case 4), we have the contradiction.

Case 6. Let $1 < \alpha$, $0 < \beta < 1$, $0 < \gamma < \max\left\{0, \frac{\alpha\beta-1}{2(1-\beta)}\right\}$.

Then $0 < \gamma < \frac{\alpha\beta-1}{2(1-\beta)}$ and $2\gamma(1-\beta) < \alpha\beta - 1$ and hence $(\alpha+2\gamma)\beta - 2\gamma - 1 > 0$. Since

$$\lim_{a\to\infty} \frac{\beta b^{2\gamma(\beta-1)}}{(y^\alpha-1)^2}\left\{y^\alpha - 1 - \frac{\alpha(1-b)(a-b)}{(a-1)} - \frac{a^{2\gamma}(y^\alpha-1)^2}{a^{2\gamma}y^\alpha - b^{2\gamma}}\right\}$$
$$= \frac{\beta b^{2\gamma(\beta-1)}}{(2^\alpha-1)^2}\left\{2^\alpha - 1 - \alpha(1-b) - \frac{(2^\alpha-1)^2}{2^\alpha}\right\}$$

and

$$\lim_{a\to\infty} \frac{1}{(1-a^{-\alpha\beta}b^{\alpha\beta}y^{\alpha\beta})(a^{-2\gamma}b^{2\gamma}-y^\alpha)^2}\{a^{2\gamma(\beta-1)} - a^{-(\alpha\beta+2\gamma)}b^{(\alpha+2\gamma)\beta}\}$$
$$\times\{a^{-2\gamma\beta}b^{2\gamma\beta} - y^{\alpha\beta}\} = 0$$

because $\lim_{a\to\infty} y = 2$ by ($\sharp_2$), we have, by (iii$'$), $2^\alpha - 1 - \alpha(1-b) - \frac{(2^\alpha-1)^2}{2^\alpha} \leqslant 0$ and

$$b \leqslant \frac{\alpha - 1 + \frac{1}{2^\alpha}}{\alpha}.$$

But the range of the value of $\frac{\alpha-1+\frac{1}{2^\alpha}}{\alpha}$ for all $\alpha > 1$ is $\left(\frac{1}{2}, 1\right)$ and we need to construct the another counter example.

In our example constructed as above, let ϵ be $0 < \epsilon < \frac{(a-b)(a-1)}{1-b}$ and let $\delta = \frac{(1-b)\epsilon}{a-1}$, $x = 1$. Then clearly $B \geqslant bI$ and $A \geqslant B$.

By the same way of the construction as above, we have the following inequality:

$$\frac{\beta b^{(\alpha+2\gamma)\beta-2\gamma}}{(1-b^\alpha)^2}\left\{\frac{(b^{2\gamma}-a^{2\gamma})(1-b^\alpha)}{b^{\alpha+2\gamma}-a^{2\gamma}} - \frac{\alpha(1-b)(a-b)}{b(a-1)}\right\}$$
$$\leqslant \frac{a^{2\gamma(1-\beta)}}{(a^{\alpha\beta}-1)(b^{\alpha+2\gamma}-a^{2\gamma})^2}\{a^{(\alpha+2\gamma)\beta} - b^{(\alpha+2\gamma)\beta}\}\{b^{(\alpha+2\gamma)\beta} - a^{2\gamma\beta}\}. \tag{iv}$$

Then we have

$$\lim_{b\to 0}\frac{\beta b^{(\alpha+2\gamma)\beta-2\gamma-1}}{(1-b^{\alpha})^2}\left\{\frac{b(b^{2\gamma}-a^{2\gamma})(1-b^{\alpha})}{b^{\alpha+2\gamma}-a^{2\gamma}}-\frac{\alpha(1-b)(a-b)}{a-1}\right\}=0$$

and

$$\lim_{b\to 0}\frac{a^{2\gamma(1-\beta)}}{(a^{\alpha\beta}-1)(b^{\alpha+2\gamma}-a^{2\gamma})^2}\{a^{(\alpha+2\gamma)\beta}-b^{(\alpha+2\gamma)\beta}\}\{b^{(\alpha+2\gamma)\beta}-a^{2\gamma\beta}\}$$
$$=\frac{a^{2\gamma(1-\beta)}}{(a^{\alpha\beta}-1)a^{4\gamma}}\cdot a^{(\alpha+2\gamma)\beta}\{-a^{2\gamma\beta}\}=-\frac{a^{(\alpha+2\gamma)\beta-2\gamma}}{a^{\alpha\beta}-1}<0.$$

This contradicts (iv). □

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Further reading

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